Reduced RLT constraints for polynomial programming

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Abstract. An extension of the reduced Reformulation-Linearization Technique constraints from quadratic to general polynomial programming problems with linear equality constraints is presented and a strategy to improve the associated convex relaxation is proposed.

1. Introduction

Reduced RLT constraints (rRLT) are a special class of Reformulation-Linearization Technique (RLT) constraints, that apply to nonconvex (both continuous and mixed-integer) quadratic programming problems subject to linear equality constraints [2, 4, 3]. rRLT are obtained by replacing some of the quadratic terms with suitable linear constraints. These turn out to be a subset of the RLT constraints for quadratic programming [7].

We present an extension of the rRLT theory to the case of general polynomial programs. Then, we show a strategy to choose the basis of a matrix involved in the rRLT constraints generation so as to tighten the bound of the associated convex relaxation. This allows to improve the performance of a spatial Branch-and-Bound algorithm applied to nonconvex NLP and MINLP problems where such convex relaxation is computed at each node.

2. Extending rRLT to polynomial programs

Let n be the number of variables, q the degree of the polynomials in the targeted problem and $\mathcal{N} = \{1, \ldots, n\}$, $Q = \{2, \ldots, q\}$. For each monomial $x_{j_1} \cdots x_{j_p}$, $p \in Q$, appearing in the problem, we define a finite sequence $J = (j_1, \ldots, j_p)$ and consider defining constraints of the following form:

$$w_J = \prod_{\ell \le |J|} x_{j_\ell} \tag{2.1}$$

(for $|J| = 1$, i.e. $J = (j)$, we also define $w_J = x_j$). For all $p \in Q$, $J \in \mathcal{N}^p$ and any permutation π in the symmetric group S_p we have that $w_J = w_{\pi J}$ by commutativity. We therefore define an equivalence relation $\sim$ on $\mathcal{N}^p$ stating that for $J, K \in \mathcal{N}^p$, $J \sim K$ only if $\exists \pi \in S_p$ such that $J = \pi K$. We then consider the index tuple set $\bar{\mathcal{N}}^p = \mathcal{N}^p/\!\sim$ to quantify over when indexing variables w_J.

We multiply the original linear constraints $Ax = b$ by all monomials $\prod_{\ell \le p-1} x_{j_\ell}$ and replace them by the corresponding added variables $w_{(J',j)}$, where $J' \in \bar{\mathcal{N}}^{p-1}$. This yields the following rRLTS:

$$\forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} \quad A\, \mathbf{w}_{J'} = b w_{J'}, \tag{2.2}$$

where $\mathbf{w}_{J'} = (w_{(J',1)}, \ldots, w_{(J',n)})$. We then consider the companion system:

$$\forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} \quad A\, \mathbf{z}_{J'} = 0. \tag{2.3}$$

Since (2.3) is a linear homogeneous system, there is a matrix M such that the companion system is equivalent to $Mz = 0$, the columns of which are indexed by sequences in $\bar{\mathcal{N}}^p$. We let $B \subseteq \bar{\mathcal{N}}^p$

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and $N \subseteq \bar{\mathcal{N}}^p$ be index sets for basic and nonbasic columns of M. We define the following sets:

$$\begin{aligned} C &= \{(x,w) \mid Ax = b \wedge \forall p \in Q, J \in \bar{\mathcal{N}}^p (w_J = \prod_{\ell \leq |J|} x_{j_\ell})\} \\ R_N &= \{(x,w) \mid Ax = b \wedge \forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} (A\,\mathbf{w}_{J'} = b w_{J'}) \wedge \\ &\quad \forall J \in N (w_J = \prod_{\ell \leq |J|} x_{j_\ell})\}. \end{aligned}$$

Theorem 1. *For each partition B, N into basic and nonbasic column indices for the companion system $Mz = 0$, we have $C = R_N$.*

3. Tightening the convex relaxation

Replacing C with R_N for some nonbasis N effectively replaces some monomial terms with linear constraints, and therefore contributes to simplify the problem. A convex relaxation for the reformulated problem is readily obtained by applying monomial convexification methods in the literature [5, 6, 1]. We observe that for any given linear system there is in general more than one way to partition the variables in basics and nonbasics. Hence the set B can be chosen in such a way as to decrease the discrepancy between the feasible region and its convex relaxation. Given $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and the sets $S = \{(x,w) \mid w = f(x)\}$ and $\bar{S} = \{(x,w) \mid \underline{f}(x) \leq w \leq \bar{f}(x)\}$, where $\underline{f}(x)$, $\bar{f}(x)$ are respectively a convex lower and a concave upper bounding function for f (and hence $\bar{S}$ is a convex relaxation of S), the *convexity gap* between S and $\bar{S}$ can be defined as the volume $V(S)$ of the set $\bar{S}$. Explicit expressions of $V(S)$ can be derived for a quadratic term x_i^2, for a bilinear term $x_i x_j$ using the Cayley-Menger formula in 3 dimensions, and for a general monomial, exploiting associativity recursively to rewrite it as product of lower degree monomials and using the preceding results.

Let B, N be the basic/nonbasic sets of column indices of the companion system, which we can write as $M_B z_B + M_N z_N = 0$. The elements of B, N are sequences $J \in \mathscr{M}$. For $S \subseteq \mathscr{M}$ and $p \in Q$ we define $V^{S,p} = \sum_{\substack{J \in S \\ |J| = p}} V_J$ and $V^S = \sum_{p \in Q} V^{S,p}$. If, for all $p \in Q$, $V^{N,p} < V^{\beta,p}$ then the total convexity gap of R_N is smaller than that of C. Thus, we aim to find N such that $V^{N,p}$ is minimized, or equivalently, to find B such that $V^{B,p}$ is maximized for all $p \in Q$. This yields the multi-objective problem:

$$\left.\begin{array}{r} \forall p \in Q \quad \max V^{B,p} \\ M_B \text{ is a basis of (2.3)} \end{array}\right\} \tag{3.1}$$

It can be shown that (3.1) is equivalent to a single-objective problem: any solution B of (3.1) maximizing V^B also maximizes $V^{B,p}$ for all $p \in Q$. In this way, we have derived a technique to choose a good basis for the companion system so as to improve the chances of tightening the lower bound of the convex relaxation associated to rRLT.

Preliminary computational experiments carried out on a set of randomly generated instances of the convex Quadratic Knapsack Problem (cQKP) show that the proposed strategy is promising in improving performances of a spatial Branch-and-Bound algorithm.

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